Painlevé type reductions in the non-Abelian Toda lattice

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Introduction

Non-Abelian analogs are known for all equations P_1–P_6, but their classification is far from complete.

Some obvious difficulties:

- for a given scalar equation, several analogs may exist with different structure of terms;
- equations may contain additional parameters, possibly non-Abelian;
- lowering of order by integration may be not possible;
- a coupled system of two first order equations may be not equivalent to a second order equation.

We demonstrate by several examples that these effects are quite common for non-Abelian Painlevé equations.

Three non-Abelian versions of P_2

The following equations pass the Painlevé–Kovalevskaya test:

$$y'' = 2y^3 + zy + by + yb + \alpha, \qquad \mathrm{P}_2^0$$

$$y'' = \pm[y, y'] + 2y^3 + zy + a, \qquad \mathrm{P}_2^1$$

$$y'' = \pm 2[y, y'] + 2y^3 + zy + by + yb + a, \quad [b, a] = \pm 2b, \qquad \mathrm{P}_2^2$$

where $y = y(z)$, a, $b \in \mathcal{A}$ (free associative algebra or square matrices of any size) and $z, \alpha \in \mathbb{C}$.

P_2^0 with $b = 0$: [Balandin & Sokolov 1998]

P_2^0 with $b \neq 0$: [Retakh & Rubtsov 2010]

P_2^1, P_2^2: [Adler & Sokolov 2021] by applying the PK test to the family

$$y'' = \kappa[y, y'] + 2y^3 + zy + b_1 y + y b_2 + a, \quad a, b_1, b_2 \in \mathcal{A}, \quad \kappa \in \mathbb{C}. \qquad (1)$$

However, we cannot guarantee that the obtained list is exhaustive. For instance, even the linear term can be generalized in many ways:

$$zy \to zy + b_1 y c_1 + b_2 y c_2 + \cdots + b_n y c_n, \quad b_i, c_i \in \mathcal{A},$$

not saying about the terms of higher degrees. It is hardly possible to analyse all such generalizations. In principle, there may exist some another integrable case beyond the (1) family.

The PK test becomes much more complicated compared to the scalar case (in addition to increase of the number of terms, we have to analyse the block structure of the matrices). Nevertheless, it remains rather effective if we restrict ourselves to some reasonable family of equations.

For instance, a large family of P_4 analogs was classified recently by Bobrova & Sokolov, with even more rich answer than for P_2.

Another way to obtain new examples is by group-invariant reductions from integrable non-Abelian PDEs.

Remark: the mirror transformation

It is worth noticing that, according to the PK test, the parameter κ in (1) is quantized, taking the values $0, \pm 1, \pm 2$.

On the other hand, there exist a transformation which changes it arbitrarily, in the case of equations which are invariant under the group

$$y \mapsto cyc^{-1}, \quad c \in \mathcal{A}$$

(this means that all parameters should be scalar; we call such equations GL-invariant).

Proposition [Golubchik & Sokolov 1997]

Let $\kappa \in \mathbb{C}$, $\kappa \neq 0$, then the general solutions of GL-invariant equations

$$y'' = \kappa[y, y'] + f(z, y, y') \qquad \text{and} \qquad \tilde{y}'' = f(z, \tilde{y}, \tilde{y}')$$

are related by the transformation

$$\kappa y = w' w^{-1}, \qquad \kappa \tilde{y} = w^{-1} w'.$$

Proof:

$$\tilde{y} = w^{-1} y w, \qquad \tilde{y}' = w^{-1} y' w + [\tilde{y}, w^{-1} w'] = w^{-1} y' w,$$
$$\tilde{y}'' = w^{-1} y'' w + [\tilde{y}', w^{-1} w'] = w^{-1} f(y, y', z) w + \kappa [\tilde{y}', \tilde{y}]$$

and we only have to use the GL-invariance property. □

Moreover, this transform admits a prolongation to the isomonodromic Lax pairs. Let the equation for y be equivalent to the compatibility condition

$$\Psi_\zeta = A\Psi, \quad \Psi' = B\Psi \quad \Rightarrow \quad A' = B_\zeta + [B, A],$$

where A and B are rational with respect to y, y' and z, with scalar coefficients. Then the equation for $\tilde{y}$ also has the Lax pair, with

$$\Psi = w\tilde{\Psi}, \quad \tilde{A} = w^{-1} A w, \quad \tilde{B} = w^{-1} B w - w^{-1} w' = w^{-1} B w - \kappa \tilde{y}.$$

For example, the equation

$$y'' = \kappa[y, y'] + 2y^3 + zy + \alpha, \quad \kappa, \alpha \in \mathbb{C}$$

admits, *for any κ*, the Lax pair $A' = B_\zeta + [B, A]$ with

$$B = \begin{pmatrix} \zeta + \kappa y & y \\ y & \kappa y - \zeta \end{pmatrix}, \quad A = \begin{pmatrix} -4\zeta^2 + 2y^2 + z & -4\zeta y - 2y' - \alpha/\zeta \\ -4\zeta y + 2y' - \alpha/\zeta & 4\zeta^2 - 2y^2 - z \end{pmatrix}$$

(the scalars ζ, z and α/ζ are understood as multiples of $1 \in \mathcal{A}$).

Thus, the isomonodromic Lax pair does not guarantees the Painlevé property.

Excercise

How to solve the equation

$$y'' = [y', y]$$

(which is just $y'' = 0$ in the commutative case)? Prove that the general solution is $y = w^{-1}w'$, where $w' = (za + b)w$, $a, b \in \mathcal{A}$. But, is it possible to solve this linear equation for w explicitly?

Self-similar reductions of mKdV-1 and mKdV-2

mKdV-1 [Marchenko 1986]:

$$u_t = u_{xxx} - 3u^2u_x - 3u_xu^2 = (D_x - \mathrm{ad}\,u)(u_{xx} + [u, u_x] - 2u^3)$$

mKdV-2 [Khalilov & Khruslov 1990]

$$\begin{aligned} u_t &= u_{xxx} + 3[u, u_{xx}] - 6uu_xu - 3(u_x + u^2)c - 3c(u_x - u^2) \\ &= (D_x + \mathrm{ad}\,u)(u_{xx} + 2[u, u_x] - 2u^3 - 3cu - 3uc), \qquad c \in \mathcal{A} \end{aligned}$$

In mKdV-2, the constant c is related with the Miura map constructed by a ψ-function corresponding to the non-Abelian value of spectral parameter:

$$u = \psi^{-1}\psi_x, \quad \psi_{xx} = v\psi - \psi c.$$

For mKdV-1, the similar chnage $u = \psi_x\psi^{-1}$ is possible only for scalar c.

Self-similar reduction:

$$u=\varepsilon\tau e^{\log(\tau)d}y(z)e^{-\log(\tau)d},\quad \tau=t^{-1/3},\quad z=\varepsilon\tau x,$$

where $3\varepsilon^3=-1$ and $d\in\mathcal{A}$. For mKdV-2, we additionally assume

$$c=(\varepsilon\tau)^2e^{\log(\tau)d}c_0e^{-\log(\tau)d},\quad 2c_0+[d,c_0]=0,\quad c_0\in\mathcal{A}$$

which implies that c remains independent of τ.

Then

$$\text{mKdV-1}\quad\to\quad\Big(\frac{d}{dz}-\operatorname{ad}y\Big)(y''-[y,y']-2y^3-zy+d)=0,$$
$$\text{mKdV-2}\quad\to\quad\Big(\frac{d}{dz}+\operatorname{ad}y\Big)(y''+2[y,y']-2y^3-zy-3cy-3yc+d)=0.$$

In contrast to the scalar case, no first integrals exist, even for $d=c=0$. However, since these equations are of the form

$$J'=\pm[y,J],$$

the lowering of order is possible due to the *partial first integral* $J=\gamma\in\mathbb{C}$, which defines an invariant sub-manifold in the solution space.

Reductions of non-Abelian Volterra lattices

Two integrable non-Abelian versions of the Volterra lattice:

VL^1	$u_{n,x} = u_{n+1}u_n - u_nu_{n-1}$	[Wadati 1980; Salle 1982]
VL^2	$u_{n,x} = u^{\mathrm{T}}_{n+1}u_n - u_nu^{\mathrm{T}}_{n-1}$	[Adler 2020]

Here $^{\mathrm{T}}$ denotes the matrix transpose or a linear map $\mathcal{A} \to \mathcal{A}$ with the property $(ab)^{\mathrm{T}} = b^{\mathrm{T}}a^{\mathrm{T}}$.

VL^1 and VL^2 are related by some implicit transformation similar to the mirror map between mKdV-1 and mKdV-2.

- Instead of self-similar reductions, we obtain Painlevé-type equations as stationary equations for non-autonomous symmetries.
- The shift $n \to n+1$ provides Bäcklund transformations which, in turn are equivalent to discrete Painlevé equations.
- Non-abelian constants can be introduced by adding classical symmetry $u_{n,\tau} = [a, u_n]$, but, for simplicity, we restrist ourselves with GL-invariant equations.

Symmetries and constraints

Like for KdV, there exists an infinite algebra of flows:

$$[\partial_{t_i},\partial_{t_j}]=0,\quad [\partial_{\tau_i},\partial_{t_j}]=j\partial_{t_{j+i-1}},\quad [\partial_{\tau_i},\partial_{\tau_j}]=(j-i)\partial_{\tau_{j+i-1}},\quad i,j\ge 1.$$

We only use symmetries that involve u_{n+k} with $|k|\le 2$.

♦ the lattice itself $\partial_{t_1}=\partial_x$;

♦ the simplest higher symmetry

$$\begin{aligned}\mathrm{VL}^1:\quad u_{n,t_2}&=(u_{n+2}u_{n+1}+u_{n+1}^2+u_{n+1}u_n)u_n\\&\qquad -u_n(u_nu_{n-1}+u_{n-1}^2+u_{n-1}u_{n-2}),\\ \mathrm{VL}^2:\quad u_{n,t_2}&=(u^{\mathrm{T}}_{n+1}u_{n+2}+(u^{\mathrm{T}}_{n+1})^2+u_nu^{\mathrm{T}}_{n+1})u_n\\&\qquad -u_n(u^{\mathrm{T}}_{n-1}u_n+(u^{\mathrm{T}}_{n-1})^2+u_{n-2}u^{\mathrm{T}}_{n-1});\end{aligned}$$

♦ the classical scaling symmetry

$$u_{n,\tau_1}=u_n;$$

♦ the master-symmetry (nonlocal for VL^1, local for VL^2)

$$\mathrm{VL}^1: \quad u_{n,\tau_2} = \big(n+\tfrac{3}{2}\big)u_{n+1}u_n + u_n^2 - \big(n-\tfrac{3}{2}\big)u_nu_{n-1} + [s_n,u_n],$$
$$s_n - s_{n-1} = u_n,$$
$$\mathrm{VL}^2: \quad u_{n,\tau_2} = \big(n+\tfrac{3}{2}\big)u^{\mathrm{T}}_{n+1}u_n + u_n^2 - \big(n-\tfrac{3}{2}\big)u_nu^{\mathrm{T}}_{n-1}.$$

Any linear combination of derivations

$$\partial_t = \mu_1(x\partial_{t_2}+\partial_{\tau_2}) + \mu_2(x\partial_x+\partial_{\tau_1}) + \mu_3\partial_{t_2} + \mu_4\partial_x$$

commute with ∂_x. Therefore, the stationary equation

$$\partial_t(u_n) = 0$$

is a constraint consistent with the lattice.

Up to equivalence transformations, there are three different cases which lead to non-Abelian Painlevé equations:

$$\begin{array}{llll} & 2(x\partial_x+\partial_{\tau_1}) \quad +\partial_{t_2} & =0 & \to \quad \mathrm{dP}_1+\mathrm{P}_4\\ x\partial_{t_2}+\partial_{\tau_2} & +\mu(x\partial_x+\partial_{\tau_1}) & +\nu\partial_x=0 & \to \quad \mathrm{dP}_{34}+\mathrm{P}_5\\ x\partial_{t_2}+\partial_{\tau_2} & & +\nu\partial_x=0 & \to \quad \mathrm{dP}_{34}+\mathrm{P}_3 \end{array}$$

- In all cases, we start from some 5-point OΔE

$$f_n(u_{n-2},u_{n-1},u_n,u_{n+1},u_{n+2};x,\mu,\nu)=0.$$

- It admits a reduction of order due to partial first integrals; the final result is a 3-point discrete Painlevé equation

$$g_n(u_{n-1},u_n,u_{n+1};x,\mu,\nu,\varepsilon,\delta)=0.$$

  with additional constants $\varepsilon,\delta\in\mathbb{C}$.
- The x-dynamics is reduced to an ODE system for (u_n,u_{n+1}) which is equivalent to a continuous Painlevé equation.

Scaling reduction: $\partial_{t_2}+2(x\partial_x+\partial_{\tau_1})=0 \;\;\rightarrow\;\; \mathrm{dP}_1+\mathrm{P}_4$

$$\begin{aligned}\mathrm{VL}^1:\;\; &(u_{n+2}u_{n+1}+u_{n+1}^2+u_{n+1}u_n)u_n-u_n(u_nu_{n-1}+u_{n-1}^2+u_{n-1}u_{n-2})\\ &\quad+2x(u_{n+1}u_n-u_nu_{n-1})+2u_n=0,\end{aligned}$$

$$\begin{aligned}\mathrm{VL}^2:\;\; &(u^{\mathrm{T}}_{n+1}u_{n+2}+(u^{\mathrm{T}}_{n+1})^2+u_nu^{\mathrm{T}}_{n+1})u_n-u_n(u^{\mathrm{T}}_{n-1}u_n+(u^{\mathrm{T}}_{n-1})^2+u_{n-2}u^{\mathrm{T}}_{n-1})\\ &\quad+2x(u^{\mathrm{T}}_{n+1}u_n-u_nu^{\mathrm{T}}_{n-1})+2u_n=0.\end{aligned}$$

This can be represented as $F_{n+1}u_n-u_nF_{n-1}=0$.

The equality $F_n=0$ is a partial first integral, consistent with ∂_x due to the identities

$$\begin{aligned}F_{n,x}&=(F_{n+1}-F_n)u_n+u_n(F_n-F_{n-1}) &&\text{for } \mathrm{VL}^1,\\ F_{n,x}&=(F^{\mathrm{T}}_{n+1}+F_n)u_n-u_n(F_n+F^{\mathrm{T}}_{n-1}) &&\text{for } \mathrm{VL}^2.\end{aligned}$$

This gives two analogs of dP_1:

$$u_{n+1}u_n+u_n^2+u_nu_{n-1}+2xu_n+\gamma_n=0, \qquad \mathrm{dP}_1^1$$

$$u^{\mathrm{T}}_{n+1}u_n+u_n^2+u_nu^{\mathrm{T}}_{n-1}+2xu_n+\gamma_n=0, \qquad \mathrm{dP}_1^2$$

where $\gamma_n:=n-\nu+(-1)^n\varepsilon$.

The continuous part of dynamics is governed by P_4 for $y = u_n$:

$$y'' = \frac{1}{2}y'y^{-1}y' + \underset{\sim}{[\kappa_i y - \gamma y^{-1}, y']} + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha)y - 2\gamma^2 y^{-1}, \quad P_4^i$$

where

$$\alpha = \gamma_{n-1} - \frac{\gamma_n}{2} + 1, \quad \gamma = \frac{\gamma_n}{2}, \quad \kappa_1 = \frac{1}{2} \text{ and } \kappa_2 = -\frac{3}{2}.$$

♦ In the scalar case, this reduction was introduced in [Its, Kitaev & Fokas 1990].

♦ Another non-Abelian version of dP_1 was studied in [Cassatella-Contra, Mañas & Tempesta 2012, 2018]:

$$u_{n+1} + u_n + u_{n-1} + 2x + \gamma_n u_n^{-1} = 0.$$

♦ More general versions of P_4 were found recently by Bobrova & Sokolov.

Master-symmetry reduction:

$$x\partial_{t_2}+\partial_{\tau_2}+\mu(x\partial_x+\partial_{\tau_1})+\nu\partial_x=0 \quad\to\quad \mathrm{dP}_{34}+\mathrm{P}_5 \text{ or } \mathrm{P}_3$$

The first step is easy (like in the previous case). It brings to 4-point equations

$$\begin{aligned}\mathrm{VL}^1:\ & x(u_{n+2}u_{n+1}+u_{n+1}^2-u_n^2-u_nu_{n-1})-(2\mu x-n+\nu-\tfrac{3}{2})u_{n+1}\\ &\quad +(2\mu x-n+\nu+\tfrac{1}{2})u_n-\mu+2(-1)^n\varepsilon=0,\end{aligned}$$

$$\begin{aligned}\mathrm{VL}^2:\ & x\big(u^{\mathrm{T}}_{n+1}u_{n+2}+(u^{\mathrm{T}}_{n+1})^2-u_n^2-u_nu^{\mathrm{T}}_{n-1}\big)-(2\mu x-n+\nu-\tfrac{3}{2})u^{\mathrm{T}}_{n+1}\\ &\quad +(2\mu x-n+\nu+\tfrac{1}{2})u_n-\mu+2(-1)^n\varepsilon=0,\end{aligned}$$

where $\varepsilon\in\mathbb{C}$ is an integration constant. To obtain Painlevé equations, we need additional partial first integral.

In the scalar case, the above equation admits the integrating factor $xu_{n+1}+xu_n+n-\nu+\frac{1}{2}$ which brings to dP_{34}:

$$(z_{n+1}+z_n)(z_n+z_{n-1})=4x\frac{\mu z_n^2+2(-1)^n\varepsilon z_n+\delta}{z_n-n+\nu},\quad z_n:=2xu_n+n-\nu.$$

Non-Abelian analogs of dP_{34} are obtained as partial first integrals from the quasi-determinants of the Lax matrices.

For $\mu \neq 0$:

$$(z_{n-1}+z_n)(z_n+(-1)^n\sigma+\omega)^{-1}(z_n+z_{n+1}) \\ = 4\mu x(z_n-n+\nu)^{-1}(z_n+(-1)^n\sigma-\omega), \qquad \mathrm{dP}^1_{34}$$

$$(z^{\mathrm{T}}_{n-1}+z_n)(z_n+(-1)^n(\sigma-\omega))^{-1}(z_n+z^{\mathrm{T}}_{n+1}) \\ = 4\mu x(z_n-n+\nu)^{-1}(z_n+(-1)^n(\sigma+\omega)) \qquad \mathrm{dP}^2_{34}$$

(where $\sigma=\varepsilon/\mu$, $\omega\in\mathbb{C}$).

For $\mu = 0$:

$$\begin{cases}(z_{n+1}+z_n)(z_n-n+\nu)(z_n+z_{n-1}) = 4x(2\varepsilon z_n+\delta), & n=2k,\\ (z_n+z_{n-1})(z_{n+1}+z_n)(z_n-n+\nu) = 4x(-2\varepsilon z_n+\delta), & n=2k+1,\end{cases} \qquad \mathrm{d}\widetilde{\mathrm{P}}^1_{34}$$

$$(z^{\mathrm{T}}_{n+1}+z_n)(z_n-n+\nu)(z_n+z^{\mathrm{T}}_{n-1}) = 4x(2(-1)^n\varepsilon z_n+\delta). \qquad \mathrm{d}\widetilde{\mathrm{P}}^2_{34}$$

These discrete equations are consistent with the flow ∂_x which turns into ODE systems for $(q,p)=(z_n,z_n+z_{n+1})$ or $(z_n,z_n+z^{\mathrm{T}}_{n+1})$.

Analogs of P_5:

$$\mathrm{dP}^1_{34}\to\quad\begin{cases}2xq_x=p(q-n+\nu)-4\mu x(q+\alpha)p^{-1}(q+\beta),\\ 2xp_x=pq+qp+p-p^2+4\mu x(p-2q-\alpha-\beta),\end{cases}\qquad \mathrm{P}^1_5$$

$$\mathrm{dP}^2_{34}\to\quad\begin{cases}2xq_x=p(q-n+\nu)-4\mu x(q+\alpha)p^{-1}(q+\beta),\\ 2xp_x=2pq+p-p^2+4\mu x(p-2q-\alpha-\beta)\end{cases}\qquad \mathrm{P}^2_5$$

(in the scalar case, P_5 is satisfied by $y=1-4\mu xp^{-1}$).

Analogs of P_3:

$$\mathrm{d}\widetilde{\mathrm{P}}^1_{34}\to\quad\begin{cases}2xq_x=p(q-n+\nu)-4xp^{-1}(2\varepsilon q+\delta),\\ 2xp_x=pq+qp+p-p^2-8\varepsilon x,\end{cases}\qquad (\text{even } n)\qquad \mathrm{P}^1_3$$

$$\mathrm{d}\widetilde{\mathrm{P}}^2_{34}\to\quad\begin{cases}2xq_x=p(q-n+\nu)-4xp^{-1}(2(-1)^n\varepsilon q+\delta),\\ 2xp_x=2pq+p-p^2-8(-1)^n\varepsilon x\end{cases}\qquad \mathrm{P}^2_3$$

(in the scalar case, P_3 is satisfied by $y=p/(2\xi)$, $x=\xi^2$).

Non-Abelian (2+1)-Toda lattice

- Polynomial form [Salle, 1982]

$$f_{n,y} = p_n - p_{n+1}, \quad p_{n,x} = f_n p_n - p_n f_{n-1}, \quad f_n, p_n \in \mathcal{A} \qquad (2)$$

- Rational form ($p_n = -g_n g_{n-1}^{-1}$, $f_n = g_{n,x} g_n^{-1}$) [Mikhailov, 1981]

$$(g_{n,x} g_n^{-1})_y = g_{n+1} g_n^{-1} - g_n g_{n-1}^{-1}$$

- Scalar lattice ($g_n = e^{u_n}$) [Mikhailov, 1979]

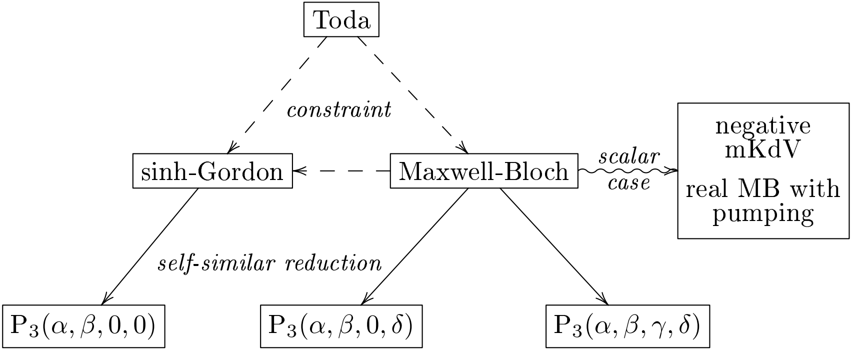
$$u_{n,xy} = e^{u_{n+1} - u_n} - e^{u_n - u_{n-1}}$$

Reductions to Painlevé equations include two steps:

$$(2) \quad \overset{constraint}{\dashrightarrow} \quad (1{+}1)\text{-equation} \quad \xrightarrow[\textit{reduction}]{\textit{self-similar}} \quad \mathrm{P}_3$$

This gives one more version of P_3, slightly different from P_3^1, P_3^2 (no classification results for P_3 are known).

Our plan in more details:



Moreover:

- ◇ the map $n \mapsto n+1$ defines Bäcklund transformations for the reduced equations
- ◇ isomonodromic Lax pairs are derived from the auxiliary linear problem for the Toda lattice

Non-Abelian sinh-Gordon equation and $P_3(1,-1,0,0)$

A simplest constraint consistent with the Toda lattice (2) is

$$f_{2n}=f,\quad f_{2n+1}=-f,\quad p_{2n}=p,\quad p_{2n+1}=p^{-1}$$
$$\Downarrow$$
$$f_y=p-p^{-1},\quad p_x=fp+pf \qquad \text{(sinh-G)}$$

(scalar equation: $f=u_x,\ p=e^{2u}\ \rightsquigarrow\ u_{xy}=e^{2u}-e^{-2u}$).

The self-similar reduction corresponds to the scaling

$$f\to\varepsilon f,\quad \partial_x\to\varepsilon\partial_x,\quad \partial_y\to\varepsilon^{-1}\partial_x.$$

As usual, we include the conjugation by arbitrary constant $a\in\mathcal{A}$:

$$z=-2xy,\quad p(x,y)=y^a p(z)y^{-a},\quad f(x,y)=-2y^{1+a}f(z)y^{-a}.$$

The result is an analog of P_3 with degenerate set of parameters $(1,-1,0,0)$:

$$zf' = \frac{1}{2}(p - p^{-1}) - f - [a,f], \quad p' = fp + pf.$$

For scalars, we have $f = p'/(2p)$ and

$$p'' = \frac{(p')^2}{p} - \frac{p'}{z} + \frac{p^2-1}{z},$$

but we cannot eliminate f in the non-Abelian case. This is possible for another version [Kawakami, 2016]

$$zQ' = 2QPQ + Q, \quad zP' = -2PQP - P + 1 - zQ^{-2}$$

(in our notation, it can be transformed to $\dots\ p' = 2fp$).

In this example, the Bäcklund transformation $n \to n+1$ is trivial: $f \to -f$, $p \to p^{-1}$.

Non-Abelian analog of the Maxwell–Bloch system

$$f_y = p - q, \quad p_x = fp + pf - \mu, \quad q_x = -fq - qf + \nu, \quad \mu, \nu \in \mathbb{C} \qquad (3)$$

♦ If $\mu = \nu = 0$ then $(pq)_x = [f, pq]$, that is, $pq = \beta(y) \in \mathbb{C}$ is a *partial* first integral. We return to sinh-Gordon by setting $pq = 1$.

♦ The Maxwell–Bloch system with the pumping parameter c reads

$$E_y = \rho, \quad \rho_x = NE, \quad 2N_x = -\rho^* E - \rho E^* + 2c, \quad \rho, E \in \mathbb{C}, \; N \in \mathbb{R}$$

[Burtsev, Zakharov & Mikhailov, 1987]. For $\rho, E \in \mathbb{R}$, we have

$$E_y = \rho, \quad \rho_x = NE, \quad N_x = -\rho E + c,$$

which is related with the scalar system (3) by the change

$$2f = iE, \quad 4p = N + i\rho, \quad 4q = N - i\rho, \quad c = 4\mu = -4\nu.$$

Self-similar reduction for this system is P_3 [Winternitz, 1992; Burtsev, 1993; Schief, 1994].

♦ For the scalar system (3), the elimination of p and q gives

$$ff_{xxy} = f_x f_{xy} + 4f^3 f_y + (\mu+\nu)f_x + 2(\nu-\mu)f^2.$$

If $\nu=\mu$, this equation is consistent with the mKdV $f_t = f_{xxx} - 6f^2 f_x$.

Thus, (3) is a non-Abelian analog of the real MB system, with additional parameter $\nu+\mu$, and of the mKdV 'negative' flow, with additional parameter $\nu-\mu$.

What is the origin of the system (3)?

Theorem. The non-Abelian Toda lattice (2) is consistent with the constraint

$$f_{n-1} + f_n = \mu_n p_n^{-1}, \quad \mu_n := \varepsilon n + \mu_0, \quad \varepsilon, \mu_0 \in \mathbb{C}. \tag{4}$$

Due to this constraint, $f=f_n$, $p=p_n$ and $q=p_{n+1}$ satisfy (3) with $\mu=\mu_n$ and $\nu=\mu_{n+1}$ for all n, and the shift $n\mapsto n+1$ is equivalent to the Bäcklund transformation

$$\tilde p = q, \quad \tilde q = p + \nu q^{-1} q_y q^{-1}, \quad \tilde f = -f + \nu q^{-1}, \quad \tilde\mu = \nu, \quad \tilde\nu = -\mu + 2\nu. \tag{5}$$

Proof. The system (3) and the map (5) are easily obtained from (2) and (4). To prove the consistency, we have to check that this map preserves this system, which is a direct calculation.

Alternatively, we notice that the constraint (4) turns the Toda lattice into a pair of $1+1$-dimensional equations: the dressing chain with zero parameters

$$f_{n,x} + f_{n+1,x} = f_n^2 - f_{n+1}^2 \tag{6}$$

and the non-autonomous Volterra-type lattice

$$f_{n,y} = \mu_n (f_{n-1} + f_n)^{-1} - \mu_{n+1} (f_n + f_{n+1})^{-1} \tag{7}$$

In this language, the consistency means that $[\partial_x, \partial_y] = 0$ which is also verified directly. ■

♦ Eq. (7) with $\varepsilon \neq 0$ is the master-symmetry for eq. (7) with $\varepsilon = 0$.

♦ In the scalar case, the consistency of (6) and (7) was observed in [Garifullin, Habibullin & Yamilov, 2015].

Zero curvature representation

The Toda lattice is the compatibility condition for the linear equations

$$\psi_{n,x} = \psi_{n+1} + f_n\psi_n, \quad \psi_{n,y} = p_n\psi_{n-1}$$

and the constraint (4) corresponds to the three-term recurrence relation

$$\psi_{n+2} = -(f_n + f_{n+1})\psi_{n+1} + (\lambda + \varepsilon y)\psi_n.$$

After some algebra, this brings to the following representations.

Theorem. The system (3) and its BT (5) are equivalent to equations

$$U_y + \frac{\nu - \mu}{2\lambda}U_\lambda = V_x + [V, U], \tag{8}$$

$$W_x = \widetilde{U}W - WU, \qquad W_y + \frac{\nu - \mu}{2\lambda}W_\lambda = \widetilde{V}W - WV, \tag{9}$$

where

$$U = \begin{pmatrix} f & \lambda \\ \lambda & -f \end{pmatrix}, \quad V = \frac{1}{\lambda}\begin{pmatrix} \frac{\mu+\nu}{2\lambda} & p \\ q & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & \lambda \\ \lambda & -\nu q^{-1} \end{pmatrix}.$$

Self-similar reductions of (3)

The scaling group + conjugation by $a \in \mathcal{A}$

$$\Downarrow$$

$$z = xy^{1/2}, \quad f = y^{1/2-a} f(z) y^a, \quad p = y^{-1/2-a} p(z) y^a, \quad q = y^{-1/2-a} q(z) y^a$$

$$\Downarrow$$

$$\begin{cases} (zf)' = 2p - 2q + 2[a, f], \\ p' = fp + pf - \mu, \\ q' = -fq - qf + \nu. \end{cases} \tag{10}$$

The BT for (10):

$$\tilde{q} = p - \frac{\nu z}{2}(fq^{-1} + q^{-1}f) - \frac{\nu}{2}q^{-1} + \frac{\nu^2 z}{2}q^{-2} + \nu[a, q^{-1}],$$
$$\tilde{p} = q, \quad \tilde{f} = -f + \nu q^{-1}, \quad \tilde{\mu} = \nu, \quad \tilde{\nu} = -\mu + 2\nu.$$

The representations (8) and (9) give, by standard procedure, the isomonodromic Lax pair for the system (10) and its BT:

$$A' = (\zeta^2 - \nu + \mu)B_\zeta + [B, A],$$

$$K' = \widetilde{B}K - KB, \quad (\zeta^2 - \nu + \mu)K_\zeta = \widetilde{A}K - KA,$$

where

$$A = \begin{pmatrix} \zeta z f - 2\zeta a - \frac{\mu+\nu}{\zeta} + \zeta\kappa & \zeta^2 z - 2p \\ \zeta^2 z - 2q & -\zeta z f - 2\zeta a + \zeta\kappa \end{pmatrix},$$

$$B = \begin{pmatrix} f & \zeta \\ \zeta & -f \end{pmatrix}, \quad K = \begin{pmatrix} 0 & \zeta \\ \zeta & -\nu q^{-1} \end{pmatrix},$$

$\kappa \in \mathbb{C}$ is an additional parameter and $\tilde{\kappa} = \kappa + 1$.

In order to identify the system (10) with P_3, we have to lower its order.

The case $\nu = \mu$ and $\mathrm{P}_3(\alpha, \beta, 0, \delta)$

If $\nu = \mu \neq 0$ then (10) admits the invariant submanifold (partial first integral)

$$J(\kappa) = 2pq - \mu(zf - 2a - \kappa) = 0.$$

This follows from the identities (easy to check)

$$J' = [f, J] \quad \text{and} \quad \tilde{J}(\tilde{\kappa}) = qJ(\kappa)q^{-1}.$$

Under the additional constraint $J = 0$, we have

$$2q = \mu p^{-1}(zf - 2a - \kappa)$$

and the system (10) takes the form

$$\begin{cases} (zf)' = 2p - \mu p^{-1}(zf - 2a - \kappa) + 2[a, f], \\ \quad p' = fp + pf - \mu. \end{cases}$$

In the scalar case, elimination of f gives P_3

$$p'' = \frac{(p')^2}{p} - \frac{p'}{z} + \frac{1}{z}(\alpha p^2 + \beta) + \gamma p^3 + \frac{\delta}{p}$$

with parameters

$$\alpha = 4, \quad \beta = \mu(4a - 1 + 2\kappa), \quad \gamma = 0, \quad \delta = -\mu^2.$$

The case $\nu - \mu = \varepsilon \neq 0$ and $\mathrm{P}_3(\alpha, \beta, \gamma, \delta)$

In this case the invariant submanifold is given by equation

$$J = 2q - \varepsilon z + \varepsilon(zf + 2a - \kappa)(2p - \varepsilon z)^{-1}(zf - 2a + \kappa - 2\mu/\varepsilon - 1) = 0.$$

It is quite difficult to see it immediately, but we can make use of the Lax representation. At the point $\zeta = \varepsilon^{1/2}$, the matrix A satisfies the equations $A' = [B, A]$ and $\widetilde{A}K = KA$ and it is easy to prove that its quasi-determinant

$$|A|_{12} = a_{12} - a_{11}a_{21}^{-1}a_{22}$$

is a partial first integral, with respect to the continuous and the discrete dynamics. The resulting system is

$$\begin{cases} (zf)' = 2p - \varepsilon z + 2[a, f] \\ \qquad\qquad + \varepsilon(zf + 2a - \kappa)(2p - \varepsilon z)^{-1}(zf - 2a + \kappa - 2\mu/\varepsilon - 1), \\ \quad p' = fp + pf - \mu. \end{cases}$$

In the scalar case, it is equivalent to P_3 with a generic set of parameters, under the change

$$w = \frac{2p(2p - \varepsilon z)}{zp' - 2\kappa p + \mu z}.$$